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THESIS

PLANARITY IN ROMDD'S OF MULTIPLE-VALUED SYMMETRIC FUNCTIONS

by

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March, 1996

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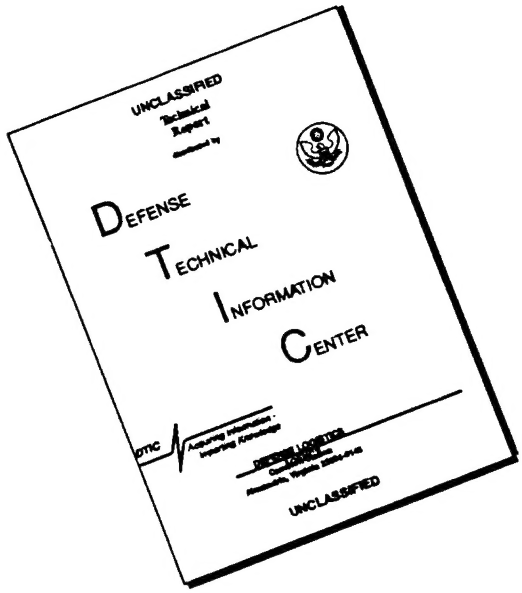

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13. ABSTRACT

An important consideration in the design of digital circuits is delay. A major source of delay in VLSI is interconnect. Crossings among interconnect require via's which cause resistance and additional delay. This thesis focuses on circuit design based on the reduced ordered multiple-valued decision diagram (ROMDD), a graph representation of a logic function. Crossings among edges in the ROMDD result in crossings in the circuit. Thus, ROMDD's without crossings reduce delay.

Since symmetric functions are important in the design of logic circuits, they are considered here. It is shown that a multiple-valued symmetric function has a planar ROMDD if and only if it is a pseudo-voting function. It is also shown that the number of such functions is $(r-1)\binom{n+r}{n+1}$, where r is the number of logic values and n is the number of variables.

It follows from this that the fraction of symmetric multiple-valued functions that have planar ROMDD's approaches 0 as n approaches infinity. Further, for planar ROMDD's of symmetric functions, it is shown that the worst case number of nodes is $n^2\left(\frac{1}{2}-\frac{1}{2r}\right)$ and the average number of nodes is $n^2\left(\frac{1}{2}-\frac{1}{(r+1)}\right)$, when n is large.

Additionally, multiple-valued *Fibonacci* functions are examined and conditions for planarity in their ROMDD representations are established.

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PLANARITY IN ROMDD'S OF MULTIPLE-VALUED SYMMETRIC FUNCTIONS

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ABSTRACT

It is shown that a multiple-valued symmetric function has a planar ROMDD (reduced ordered multiple-valued decision diagram) if and only if it is a pseudo-voting function. It is also shown that the number of such functions is $(r-1)\binom{n+r}{n+1}$, where r is the number of logic values and n is the number of variables.

It follows from this that the fraction of symmetric multiple-valued functions that have planar ROMDD's approaches 0 as n approaches infinity. Further, for planar ROMDD's of symmetric functions, it is shown that the worst case number of nodes is $n^2\left(\frac{1}{2}-\frac{1}{2r}\right)$ and the average number of nodes is $n^2\left(\frac{1}{2}-\frac{1}{(r+1)}\right)$, when n is large.

Additionally, multiple-valued *Fibonacci* functions are examined and conditions for planarity in their ROMDD representations are established.

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LIST OF VARIABLES AND ABBREVIATIONS

α . . . . A logic value in $\{0, 1, ..., a\}$.

a . . . . A logic value that determines a partition of edge values of a node.

A . . . . An assignment of values to the variables of a function.

$A_r(n)$ . . . . The average number of nodes in ROMDD's of pseudo-voting functions.

β . . . . A logic value in $\{a+1, a+2, ..., r-1\}$.

B . . . . An assignment of values to the variables of a function.

f . . . . A function.

F_i . . . . The i^{th} Fibonacci number or weight of a Fibonacci threshold function.

i . . . . An index or logic value.

j . . . . An index or logic value.

m . . . . Number of nodes, number of specific logic values, or a secondary index.

M . . . . The upper partition of logic values determined by a: $\{a+1, a+2, ..., r-1\}$.

$M_{\text{pseudo-voting}}$ . . . . . The number of pseudo-voting functions.

n . . . . Number of nodes or an index.

η . . . . A node of a decision diagram.

$n_a(A)$ . . . . The number of variables whose value is in M.

N_{complete} . . . . The total number of nodes in complete symmetric decision diagrams of pseudo-voting functions.

$N_{\text{reduction}}$ . . . . The reduction of nodes that occurs because of consecutive logic values on terminal nodes.

N_T . . . . The total number of nodes before reduction.

LIST OF VARIABLES AND ABBREVIATIONS (cont'd)

r .... The radix of a function(number of possible logic values for each variable, e.g. 0, 1, ..., r-1).

R_i .... The reduction of nodes with respect to logic value i.

R_j .... The reduction of nodes with respect to logic value j.

R_T .... The total reduction of nodes.

$R_{T\text{-}WC}$ .... The total reduction of nodes for the worst case.

T_i .... The threshold value that must be met for $f = i$.

$WC_r(n)$ .... The worst case number of nodes in ROMDD's of pseudo-voting functions.

x_i .... A function variable associated with the i^{th} level of a decision diagram.

ALU .... Arithmetic and Logic Unit

BDD .... Binary Decision Diagram

CW .... Cumulative Weight

FPGA .... Field Programmable Gate Array

MDD .... Multiple-valued Decision Diagram

MVL .... Multiple-Valued Logic

MWS .... Maximum Weighted Sum

OMDD .... Ordered Multiple-valued Decision Diagram

ROBDD ...... Reduced Ordered Binary Decision Diagram

ROMDD ...... Reduced Ordered Multiple-valued Decision Diagram

VLSI .... Very Large Scale Integration

I. INTRODUCTION

Conventional computers use the binary number system, which is based upon two levels of logic. Computers in the 1940's used relays, which had two stable states, open and closed. Tubes and transistors have two stable states, saturation (conducting) and cutoff (nonconducting). In conventional VLSI circuits, these two levels are encoded as voltage, where 0.0 volts represents a logic 0 and 2.5 to 5.0 volts represents a logic 1. The restriction of two logic levels applies throughout the circuit.

Two logic levels naturally make a binary number system a sensible choice for digital computers based on conventional VLSI. However, one disadvantage of the binary number system is that numbers require many bits to be represented as binary. For example, the decimal number 2048 is represented by the 12 bit binary number 100000000000. A decimal number exceeding one million requires at least 20 bits to be represented in the binary number system.

There are also significant disadvantages to binary in implementation. The majority of VLSI chip area is devoted to *interconnect*, i.e. bus lines. Interconnect occupies physical area even when not in use. Additionally, the insulation between the wires used for interconnect also requires area on the chip. All this area is physical space that cannot be devoted to devices. Two levels of logic also place a burden upon chip connecting pins that must maintain a minimum size and thickness for strength and reliability. This is referred to as the *pinout problem*. In binary ALU operations, limits are imposed on the speed of arithmetic circuits due to the *carry* (borrow) between digits.

The disadvantages of a binary number system are reduced when a *multiple-valued logic* (MVL) number system is implemented. Fewer bits are needed to represent numbers and more efficient use is made of interconnect when more than two levels of logic are implemented. For example, in a four-valued number system, a single digit may represent four logic values (0, 1, 2, 3). The same information representation would require two bits in binary, with $0_4 = 00_2$, $1_4 = 01_2$, $2_4 = 10_2$, and $3_4 = 11_2$. Therefore, from a physical point of view, a wire in a four-valued system would carry twice the information of a binary system. This would reduce the required chip area for interconnect by one half. There would also be a savings in chip area from a reduction in insulation because one half of the area that was devoted to insulation between binary wires would no longer be needed with four-valued wires. [Ref. 1]

A binary number system presents similar difficulties in representing binary logic (Boolean) functions by truth table because the number of bits required increases at an exponential rate in relation to the number of function variables. Because of this, a more efficient, graphical method of representing Boolean functions has been developed. For more than a decade, binary decision diagrams (BDD's) have been used to efficiently represent binary (switching) functions. Introduced by Lee [Ref. 2] in 1959, and further developed by Akers [Ref. 3] in 1978, it was not until 1986 with a paper by Bryant [Ref. 4] that BDD's have become a predominant data structure for switching function representation.

The classical representations such as truth tables and Karnaugh maps prove to be impractical for large functions as their size increases on the order of $O(2^n)$ where n is the number of function variables or arguments. The worst case complexity of a BDD, for

symmetric functions, has been documented as $O(n^2)$. [Refs. 4,5]

To construct the BDD for a given function $f(x_1, x_2, \ldots, x_n)$, a *root node* is used to represent the function itself, and two *children* nodes are attached representing the subfunctions, $f(1, x_2, \ldots, x_n)$ and $f(0, x_2, \ldots, x_n)$. To each of these children, two more children are attached to represent the assignments to x_2, and this is continued until all variables are assigned. Each node represents the *Shannon's expansion* of the Boolean function, $f = (\bar{x}_i \cdot f_0) \vee (x_i \cdot f_1)$, where i is the index of the node and f_0 and f_1 are the functions of the nodes pointed by the 0- and 1-edges [Refs. 6,7]. The *terminal* nodes represent 0 and 1, the only functions independent of all variables. Whenever the same subfunction appears in different parts of the diagram, all instances are converged into one node. Also, nodes with two identical children are deleted. A BDD representing the function, $f = x_1x_2 + x_3x_4$ is shown in Fig. 1.

As previously discussed, multiple-valued logic exhibits several advantages over binary. Multiple-valued logic functions can be represented by multiple-valued decision diagrams (MDD's) which are a natural extension of BDD's. MDD's have been treated by Miller [Ref. 8] and Sasao [Ref. 9]. This thesis expands upon a preliminary version that has been accepted for publication [Ref. 10]. It is also an extension of the results on *planar* MDD's as described in Sasao and Butler [Ref. 11].

Two types of functions are considered. In the first type, a *multiple-valued* function, $f: R^n \rightarrow R$, where $R = \{0,1,\ldots,r\text{-}1\}$, both the function and the variables take on values from R. We denote a function with $r = 2$ as a *switching* function. In the second type, $f: R^n \rightarrow \{0,1\}$, the function is two-valued, and the variables are r-valued.

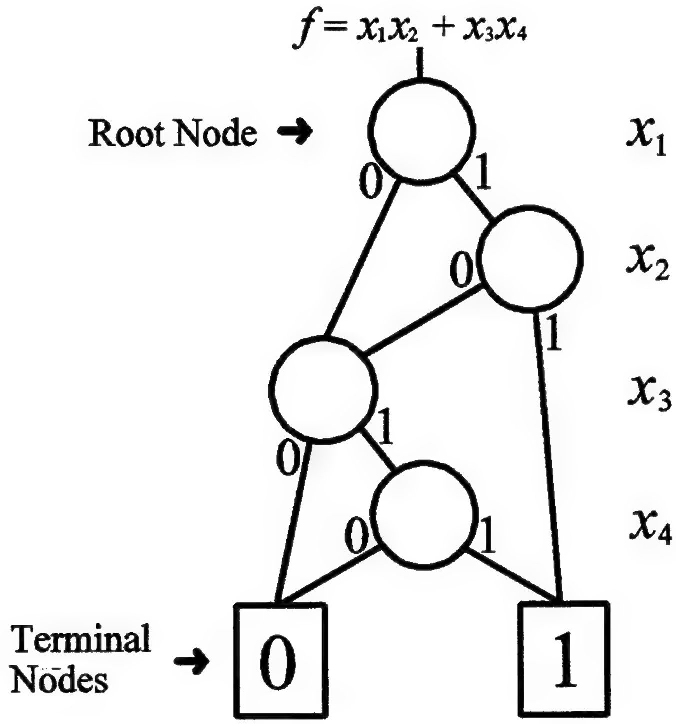


Figure 1. A binary decision diagram(BDD) representation of $f = x_1x_2 + x_3x_4$.

An MDD of a function $f(x_1, x_2, ..., x_n)$ is a directed graph that has a root node (i.e., no incoming edges) which represents f. From this node, there are r outgoing edges labeled 0, 1, ..., and r-1 directed to nodes that represent $f(0, x_2, ..., x_n)$, $f(1, x_2, ..., x_n)$, ..., and $f(r\text{-}1, x_2, ..., x_n)$, respectively. For each of these nodes, there are r outgoing edges that go to nodes that have r outgoing edges, etc. A terminal node is a node with no outgoing edges. It is labeled by 0, 1, ..., or r-1, and corresponds to a logic value of the function. An MDD is a data structure. To reduce storage requirements, the following rules are applied.

Merging Rule If two nodes η_1 and η_2 represent the same function, they are combined into one, as are descendent nodes and edges.

Elimination Rule If a node η_1 has all descendants going to the same node η_2, then η_1 is eliminated and all incoming edges to η_1 go to η_2.

Definition 1 *An ordered multiple-valued decision diagram (OMDD) is an MDD in which the relative order of any pair of variables is the same for all paths from the root node to any terminal node.*

Definition 2 *A reduced OMDD or ROMDD is an OMDD in which the merging and elimination rules have been applied to the greatest extent possible.*

Figure 2 shows the ROMDD representation of the function in Fig. 1 as $f = X_1 + X_2$ where $X_1(X_2)$ = 0, 1, 2, and 3 when $x_1x_2(x_3x_4)$ = 00, 01, 10, and 11, respectively. Notice the reduction in nodes from Fig. 1 which is achieved by using multiple-valued logic with $r = 4$. The function of Fig. 2 only requires two variables ($n = 2$), and thus its ROMDD provides a more compact representation over the ROBDD.

Bryant [Ref. 4] has shown that, for any given ordering of variables, the ROBDD is unique. Therefore, regardless of what order the merging and elimination rules are applied, the final ROBDD is the same. The same argument applies to ROMDD's.

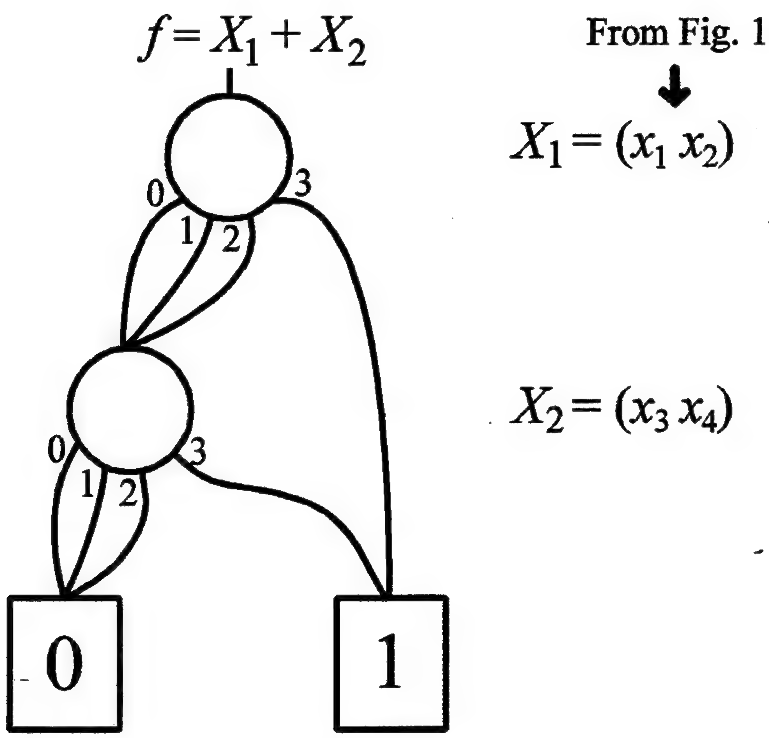


Figure 2. The ROMDD representation of the function in Fig. 1.

A special type of OMDD is examined in this thesis. As discussed previously, in VLSI, a significant source of delay is interconnect, and a significant component of interconnect delay occurs at *crossings*. For example, in field programmable gate arrays (FPGA's), a significant source of delay occurs in crossings among interconnections between cells. Via resistance, and thus delay, increases as feature size is decreased. For a discussion of circuit implementations based on MDD's and the role of crossings in such realizations, the reader is referred to [Refs. 11,12]. The restrictions in [Refs. 11,12] are adopted in this thesis and restated as follows.

Restriction 1

a: All edges are directed downward throughout their length,

b: All edges emerging from a node are labeled 0, 1, ..., r-1 *from left to right, and*

c: The terminal nodes (representing constant functions) are labeled 0, 1, ..., r-1 *from left to right.*

Restriction 1(a) precludes, for example, arcs (edges) that extend *around* the root node or terminal nodes (e.g. Fig. 13 of [Ref. 11]). It is a simplifying assumption that makes uniform the interconnection between levels. Restriction 1(b) and 1(c) are also simplifying assumptions. However, they can be removed, enlarging the set of functions for which the results apply. For now, these restrictions allow a tractable analysis.

Definition 3 *An OMDD is planar if it can be drawn without crossings.*

Because of their importance in logic design, we consider symmetric functions. Symmetric functions are indispensable in arithmetic circuits; indeed, such circuits represent one of the most important applications of multiple-valued logic [Ref. 13].

Definition 4 *A symmetric function is a function that is unchanged by any permutation of variables.*

In this thesis, multiple-valued functions and their representation using decision diagrams are considered. Necessary and sufficient conditions for planarity in the ROMDD's of symmetric functions is shown.

II. BACKGROUND

In this chapter, conditions that cause *non*-planarity in ROMDD's are considered.

Lemma 1 *If the ROMDD of a multiple-valued variable, two-valued function has at least two nodes associated with the lowest variable, then it is non-planar.*

Proof Assume x_n labels the variable just above the terminal nodes. Consider a node η at the x_n level. Because of the elimination rule, not all of its edges go to 0 and not all go to 1. For there to be no crossings among edges from η to 0 and 1, the $x_n = 0$ edge must go to the terminal node 0 and the $x_n = r - 1$ edge must go to the terminal node 1. That is, if all edges of n go to one node, then n is eliminated by the elimination rule. Since there are two nodes at the x_n level, each satisfying this requirement, there is at least one crossing, as shown in Fig. 3.

Q.E.D.

This result allows one to make the following observation.

Definition 5 *f is a voting function with $f = j$ iff* $T_j \leq \sum_{i=1}^{n} x_i < T_{j+1}$, *where* $0 = T_0 \leq T_1 \leq \ldots \leq T_{r-1} \leq T_r = n(r - 1) + 1$. *$g$ is a binary voting function on multiple-valued variables if it is a voting function with* $T_2 = T_3 = \ldots = T_r = n(r - 1) + 1$. *Associated with g is a weight-threshold vector* $(1,1,\ldots,1;T)$, *where* $T = T_1$.

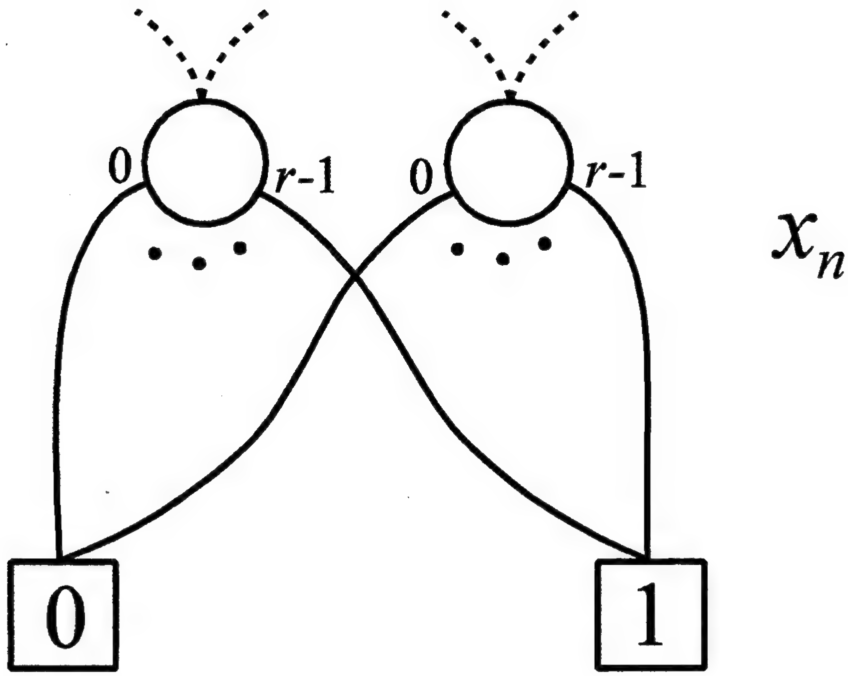


Figure 3. An ROMDD with at least one unavoidable crossing at the x_n level.

Lemma 2 *For any n* > 1 *and r* > 2, *there exists a function f with an ROMDD which is not planar for any ordering of the variables.*

Proof Consider a binary voting function *f* on multiple-valued variables, with weight-threshold vector (1,1,...,1;2). Fig. 4 shows the nodes associated with the last variable in the ordering. There are two, one that can be reached with a cumulative weight (*CW*) of 0 and the other with *CW* = 1. Note that there are *r* - 2 unavoidable crossings. Since *f* is totally symmetric, altering the variable order will not change the ROMDD.

Q.E.D.

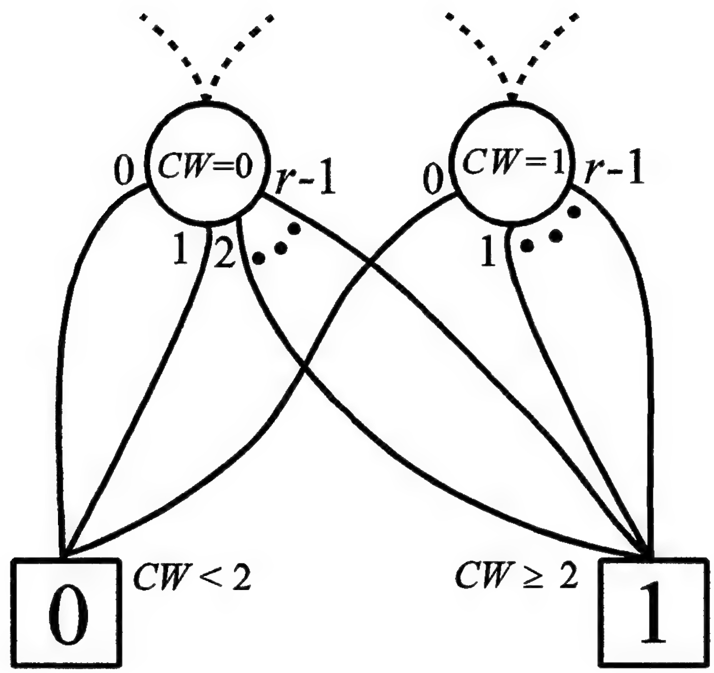


Figure 4. An ROMDD with r - 2 unavoidable crossings.
(CW = Cumulative Weight)

Now consider symmetric multiple-valued logic functions. A necessary and sufficient condition for planarity of ROBDD's of binary voting functions exists [Ref. 12]. This result is extended to functions with r-valued variables for $r > 2$.

Lemma 3 *Let $f(x_1, x_2, ..., x_n)$ be a binary voting function with n r-valued variables, where $n > 1$ and $r > 2$. f has a non-planar ROMDD iff f has a weight-threshold vector $(1,1,...,1;T)$, where $1 < T < n(r - 1)$.*

Proof *(if)* Let f be a symmetric threshold function with weight-threshold vector $(1,1,...,1;T)$, where $1 < T < n(r - 1)$. It is shown that this function has a non-planar ROMDD as follows.

Assume, without loss of generality, that the order of the variables from top to bottom is $x_1, x_2, ...,$ and x_n. Consider two assignments A and B of values to the upper $n - 1$ variables $x_1, x_2, ...,$ and x_{n-1} such that $\sum_{i=1}^{n-1} x_i = \max(0, T-(r-1))$ and $\sum_{i=1}^{n-1} x_i = \min((n-1)(r-1), T-1)$, respectively. Since all weights in the weight-threshold vector are 1, $\sum_{i=1}^{n-1} x_i$ is the number of variables equal to 1 in the assignments A and B.

Consider two assignments $A_{x_n=0}$ and $A_{x_n=r-1}$ to all variables $x_1, x_2, ...,$ and x_n such that $x_n = 0$ and $r - 1$ respectively, while the values assigned to $x_1, x_2, ...,$ and x_{n-1} are made according to A. Since $T > 1$ and $r > 2$, assignment $A_{x_n=0}$ results in $\sum_{i=1}^{n} x_i < T$. Therefore, $f = 0$ for $A_{x_n=0}$. However, $A_{x_n=r-1}$ results in $\sum_{i=1}^{n} x_i \geq T$. That is, if $\sum_{i=1}^{n-1} x_i = T-(r-1)$, we have $\sum_{i=1}^{n} x_i = T$, and if $\sum_{i=1}^{n-1} x_i = 0$, then $T \leq (r - 1)$, since $\sum_{i=1}^{n-1} x_i = \max(0, T-(r-1))$. It follows that $\sum_{i=1}^{n} x_i \geq T$, since $x_n = r - 1$. Therefore, $f = 1$ for $A_{x_n=r-1}$. Because the value of x_n determines

whether $f = 0$ or 1 with assignment A, it follows that A corresponds to a path to a node η_1 at the x_n level. Further, there is an edge from η_1 to 0 labled 0 and an edge from η_1 to 1 labled $r - 1$.

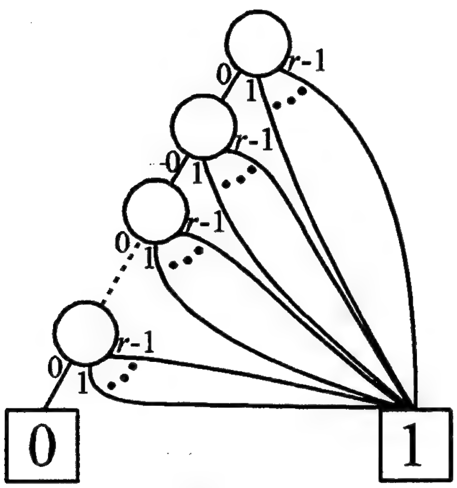
By a similar argument, it can be shown that assignment B corresponds to a path to a node η_2 with an edge labled 0 going to 0 and an edge labled $r - 1$ going to 1.

It is now shown that η_1 and η_2 are distinct nodes, by showing that the weight accumulated across x_1, x_2, ..., and x_{n-1} is different for these two nodes. For $1 < T < r$, η_1 is associated with a weight of $\max(0, T - (r - 1)) = 0$, while η_2 is associated with a weight of $\min((n - 1)(r - 1), T - 1) = T - 1 > 0$, since $n > 1$ and $T > 1$. For $r \leq T \leq (n - 1)(r - 1)$, η_1 is associated with a weight of $\max(0, T - (r - 1)) = T - (r - 1)$, while η_2 is associated with a weight of $\min((n - 1)(r - 1), T - 1) = T - 1$, which are different, since $r > 2$. For $(n - 1)(r - 1) < T < n(r - 1)$, η_1 is associated with a weight of $\max(0, T - (r - 1)) = T - (r - 1)$, while η_2 is associated with a weight of $\min((n - 1)(r - 1), T - 1) = (n - 1)(r - 1)$, which are different, since $(n - 1)(r - 1) > T - (r - 1)$ for T in this range. Thus η_1 and η_2 are distinct nodes for all T bounded by $1 < T < n(r - 1)$. Since there are two distinct nodes at the x_n level, Lemma 1 applies and one may conclude that the ROMDD for f is non-planar.

(only if) Assume that f has a non-planar ROMDD and assume on the contrary, that either $T \leq 1$ or $n(r - 1) \leq T$. If $T = 1$, then f has an ROMDD as shown in Fig. 5(a), which has no crossings, contradicting the assumption that f has a non-planar ROMDD. That is, the ROMDD for f is unique; no reordering of variables produces a different structure, specifically one with crossings.

If $T < 1$, then $f = 1$ and is represented by a single terminal node labeled 1 which is planar, contradicting the assumption. If $T = n(r - 1)$, f has the ROMDD shown in Fig. 5(b) which is planar, again contradicting the assumption. If $n(r - 1) < T$, then $f = 0$ and is represented by a single terminal node labeled 0, which is again planar. Thus, it must be that $1 < T < n(r - 1)$.

Q.E.D.


(a) (1,1,...1;1) The *OR* function

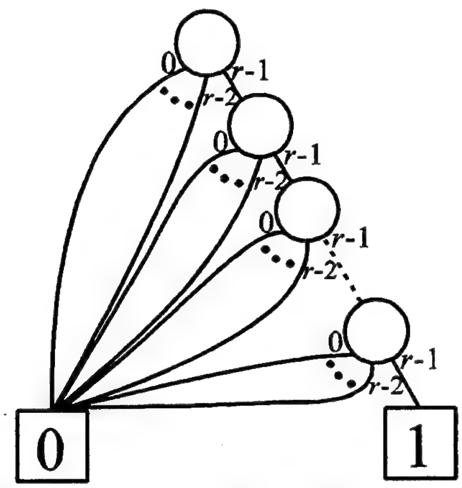

(b) (1,1,...1;n(r-1)) The *AND* function

Figure 5. Planar ROMDD's for Lemma 3.

It is interesting that Lemma 3 cannot be stated for n and r outside the range $n > 1$ and $r > 2$. That is, if $n = 1$, then all ROMDD's for f are represented by the structure shown in Fig. 6, which is planar.

Consider $r = 2$. One finds that the ROMDD for the function f associated with weight-threshold vector (1,1,1;2) as shown in Fig. 7 is planar. For this case, there exists a

weight-threshold vector (1,1,1;T) with $1 < T < n(r - 1)$ that corresponds to an ROMDD which is planar. Therefore, Lemma 3 does not apply when $r = 2$.

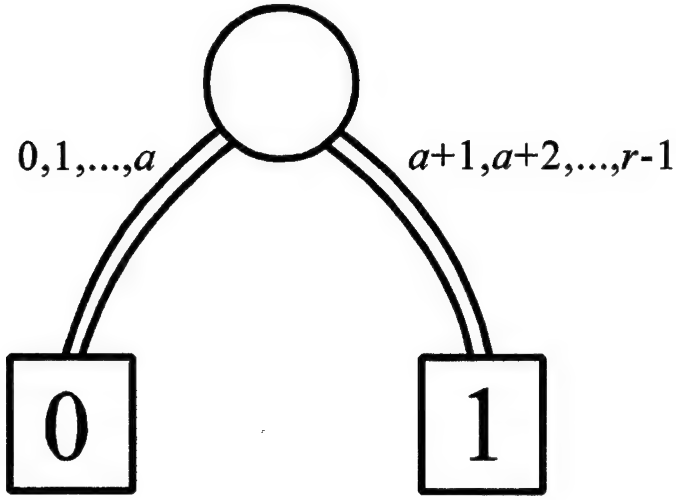


Figure 6. ROMDD structure for $n = 1$.

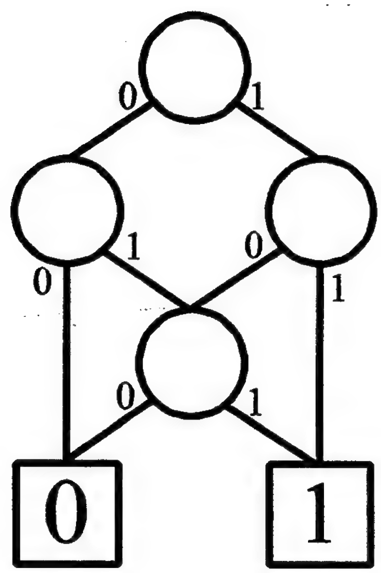


Figure 7. Planar ROMDD for T in the range $1 < T < n(r - 1)$ with $r = 2$.

III. PLANAR ROMDD'S OF SYMMETRIC FUNCTIONS

In this chapter, a necessary and sufficient condition for an r-valued symmetric function to have a planar ROMDD is shown. Such a condition has already been established for $r = 2$. Specifically,

Lemma 4 [Ref. 12] *A symmetric switching function f has a planar ROBDD iff f is a voting function.*

It is tempting to believe that this extends to multiple-valued functions. However, a counterexample exists for the same statement when the radix r exceeds 2. The function whose ROMDD is shown in Fig. 8 is symmetric and has a planar ROMDD. However, it is *not* a voting function. For example, $x_1x_2 = 11$, yields $f = 0$ while $x_1x_2 = 02$ yields $f = 1$. That is, two assignments of values to the variables with the same sum yield a different value of f. Further counterexamples are provided by Lemma 3 for the case where function *output* values are limited to 2.

Definition 6 *Let* $M = \{a+1, a+2, \ldots, r\text{-}1\}$ *be a proper subset of logic values, where* $0 \leq a \leq r - 2$. *Given an assignment A of values to the variables* $x_1, x_2, \ldots,$ *and* x_n, *let* $n_a(A)$ *be the number of variables whose value is in M. A multiple-valued function f is a pseudo-voting function if there exists a value a such that f(A) depends only on* $n_a(A)$ *and* $f(A) \geq f(A')$ *iff* $n_a(A) \geq n_a(A')$.

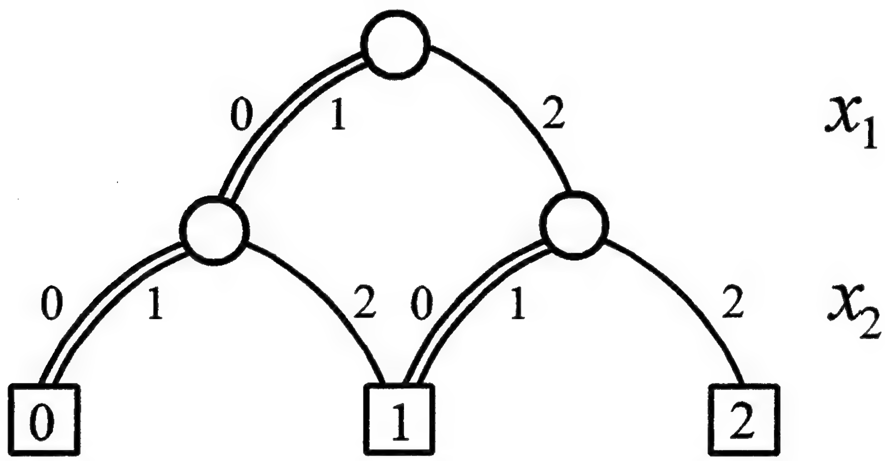


Figure 8. A counterexample to the statement that a multiple-valued function has a planar ROMDD iff it is a voting function.

In a multiple-valued pseudo-voting function, the variable values are partitioned into two parts. For some assignment A of values to these variables, a count, $n_a(A)$, is made of the number of variables that fall in the upper part of the variable logic value partition, and this determines the function value. A further restriction exists that the function value for some assignment A is never greater than for another assignment A', if $n_a(A) < n_a(A')$.

Example 1 Consider the 3-valued function shown in Fig. 9. This function is a pseudo-voting function with $a = 1$. Hatched regions show variable values in M.

Note that, when $r = 2$, a pseudo-voting function is a conventional voting function. The function in Fig. 9 has an ROMDD of the form shown in Fig. 8 above, which is planar.

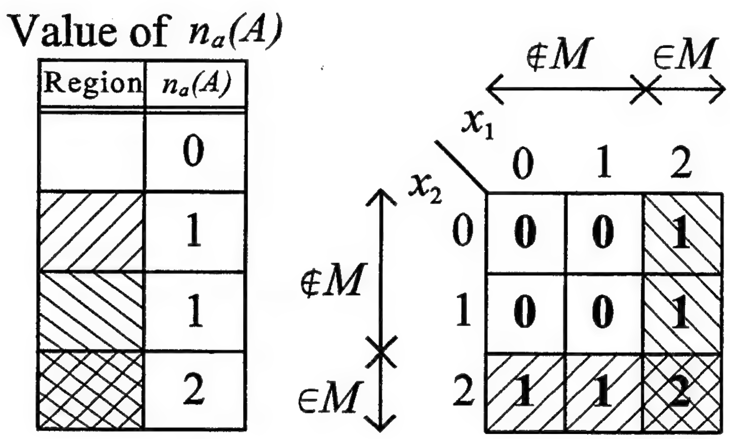


Figure 9. Map of a pseudo-voting function.

Consider f, a pseudo-voting function in which the variable values are divided into two contiguous parts, the upper part being M. Then, f is realized by the planar OMDD shown in Fig. 10 below. Here, all nodes are shown, even nodes that can be eliminated by the merging and elimination rules. Such an OMDD is called a *complete symmetric decision diagram* [Ref. 11]. A terminal node η is labeled by the number of variables that belong to M in the assignment A of values to variables that corresponds to the path from the root node to η.

The main result is,

Theorem 1 *A multiple-valued symmetric function f with $n > 1$ variables has a planar ROMDD iff f is a pseudo-voting function.*

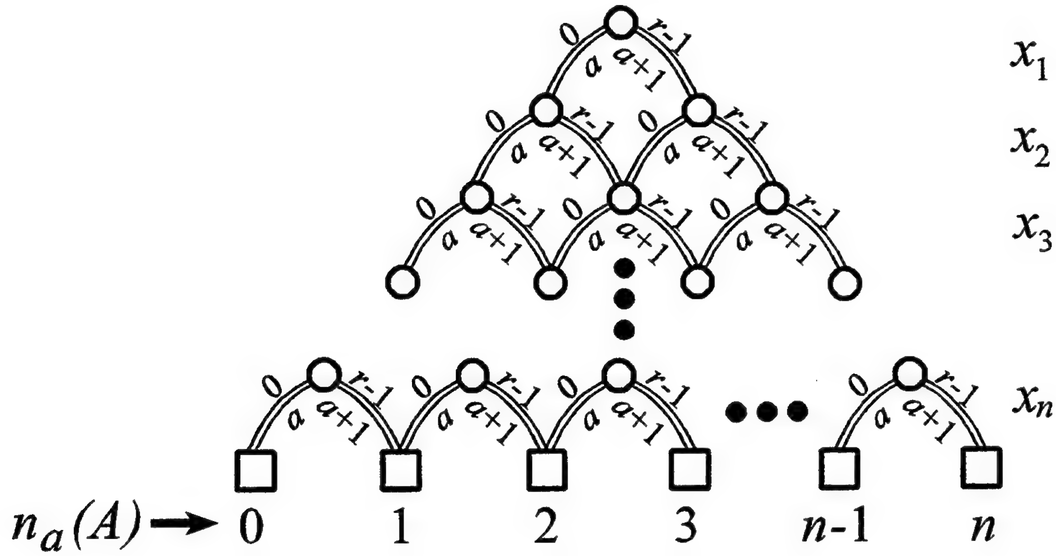


Figure 10. Complete symmetric decision diagram.

Proof *(if)* Since f is a pseudo-voting function, functional logic values labeling the terminal nodes are in ascending order left to right. We can apply the merging and elimination rules to produce an ROMDD of f. For example, two adjacent nodes labeled by the same logic value and their parent node can be replaced by a single node. Both rules preserve planarity. Since the original OMDD, as given in Fig. 10, is planar, the resulting ROMDD is also planar.

(only if) Consider a multiple-valued symmetric function f that has a planar ROMDD. First, it is shown that every node with children has exactly two children. Then, it is shown that the distribution of edges to children is the same for every node. This allows f to be realized by a complete symmetric decision diagram, as shown in Fig. 10 with the terminal nodes labeled by logic values in ascending order left to right. It can then be concluded that f is a pseudo-voting function.

Consider a node η in the ROMDD of f, as shown in Fig. 11 below.

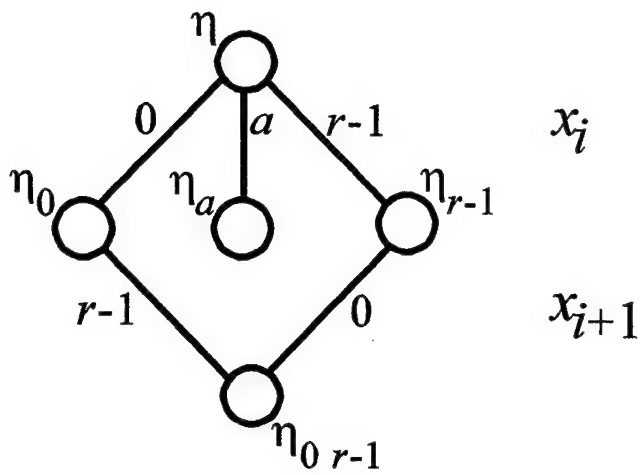


Figure 11. A node η of the ROMDD of f.

Assume that η is associated with x_i and there is at least one child of η that is associated with x_{i+1}. That is, there are at least two variables between η and a terminal node. Such a node exists because $n > 1$. Let η_0, η_a, and $\eta_{r\text{-}1}$ be the children nodes of η associated with edges labeled by 0, a, and r-1, respectively where $0 < a < r - 1$.

First, η_0 and $\eta_{r\text{-}1}$ are distinct. Indeed, if $\eta_0 = \eta_{r\text{-}1}$, then edges labeled 1, 2, ... , and r - 2 from η must also go to $\eta_0 = \eta_{r\text{-}1}$. Otherwise, there are crossings. However, by the elimination rule, η would be eliminated. Second, η_0 and $\eta_{r\text{-}1}$ are not both terminal nodes. Indeed if they were both terminal nodes, they would have to be the same node, since by symmetry of f, $x_ix_{i+1} = 0(r\text{-}1)$ and $(r\text{-}1)0$ must lead to the same node. However, as discussed earlier, η_0 and $\eta_{r\text{-}1}$ must be distinct.

Consider now the paths originating from η_a. Not all can go to $\eta_{0\ r\text{-}1}$. Otherwise, η_a does not exist by the elimination rule. But, for the edges from η_a to go to nodes outside the diamond shown in Fig. 11 above, crossings are required. It follows, therefore, that either $\eta_0 = \eta_a$ or $\eta_a = \eta_{r\text{-}1}$, regardless of the value of a. Since the planarity of the ROMDD of f excludes crossings among edges from η to its children, there exists an a such that $\eta_0 = \eta_1 =\ldots= \eta_a$ and $\eta_{a+1} = \eta_{a+2} =\ldots= \eta_{r\text{-}1}$.

It is now shown that a is the same for every node. Consider, for example, the root node and children nodes, as shown in Fig. 12 below. A claim is made that $a' = a$. On the contrary, suppose $a' \neq a$. First, suppose that $a' < a$. Then, η_4, which is reached when $x_1x_2 = a'a$, must be the *same* as η_3, which is reached when $x_1x_2 = a\,a'$, since f is symmetric. Since $\eta_3 = \eta_4$, all children nodes of η_1 are the same and, by the elimination rule, η_1 does not exist. Next, suppose that $a' > a$. Since f is symmetric, the node corresponding to $x_1x_2 = a(r\text{-}1)$ must be the same as the node corresponding to $x_1x_2 = (r\text{-}1)a$. Thus, it follows that $a'' \geq a$. Since $a' > a$ and $a'' \geq a$, the node corresponding to $x_1x_2 = a'a$ is η_4. Since $x_1x_2 = a\,a'$ corresponds to η_3 and f is symmetric, $\eta_3 = \eta_4$, and all children nodes of η_1 are the same. By the elimination rule, η_1 does not exist. From this, $a' = a$ is concluded. By a similar argument, it can be shown that $a'' = a$, and that all left-going edges of all such nodes are labeled by $\{0, 1, \ldots, a\}$. From this, it follows that all right-going edges are labeled by $\{a+1, a+2, \ldots, r\text{-}1\}$. Therefore, the function realized by the ROMDD depends not on the specific value of a variable x_i but on whether the value of x_i is in $\{0, 1, \ldots, a\}$ or in $\{a+1, a+2, \ldots, r\text{-}1\}$.

Edges from a node η can go only to the next level down (if the final value of the function is, up to this point, undetermined) or to a terminal node (if the final value of the

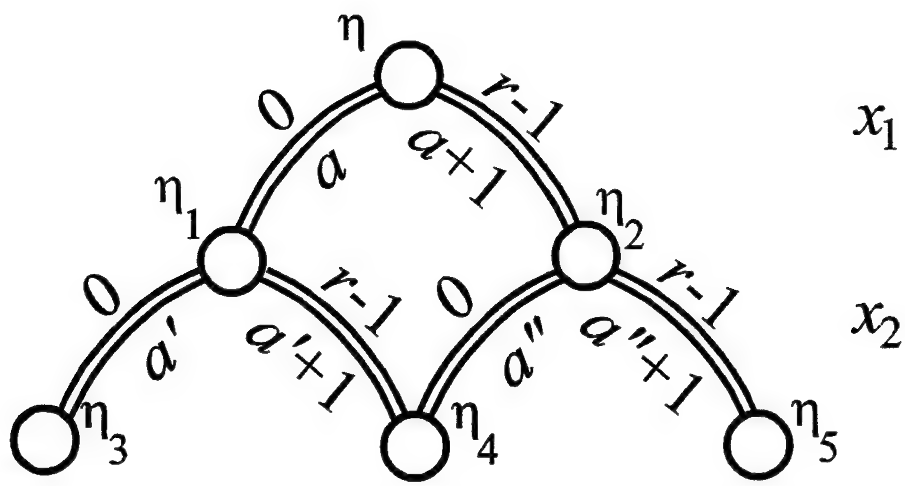


Figure 12. Root node η and its children nodes.

function is, up to this point, completely determined). That is, certain variables cannot be skipped, and others not skipped causing the realized function to be dependent on some variables and not on others, since the function is symmetric.

The OMDD (not necessarily reduced) that realizes f has the structure shown in Fig. 10. The characteristic diamond shape, as shown in Fig. 13 below occurs because the function realized when $x_i x_{i+1} = \alpha\beta$, where $\alpha \in \{0, 1, \ldots, a\}$ and $\beta \in \{a+1, a+2, \ldots, r-1\}$ is the same as the function realized when $x_i x_{i+1} = \beta\alpha$. From Restriction 1, the terminal nodes of a planar ROMDD are labeled in ascending order left to right. The function realized by this ROMDD is a pseudo-voting function.

Q.E.D.

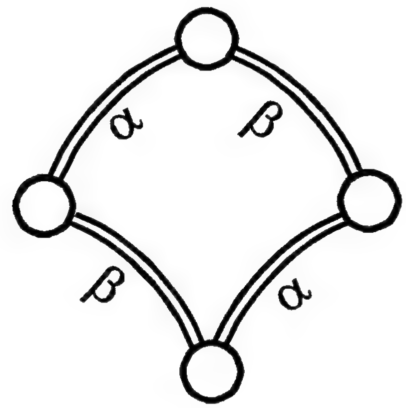


Figure 13. Characteristic diamond shape in an ROMDD of a symmetric function.

By comparing Lemma 3 with Theorem 1, it can stated,

Corollary 1 *A two-valued function with multiple-valued variables associated with weight-threshold vector* (1,1, ..., 1; *T*) *is a pseudo-voting function iff* $T = 1$ or $n(r - 1)$.

This chapter ends with counting pseudo-voting functions.

Lemma 5 *The number of pseudo-voting functions with n variables and r values is*

$$M_{\text{pseudo-voting}} = (r-1)\binom{n+r}{n+1}.$$

Proof The ways to configure a complete symmetric decision diagram of a pseudo-voting function are counted. First, there are $r - 1$ ways to partition the r logic labels of the

outgoing edges from each node. Second, the number of ways to assign logic values in ascending order left to right of the terminal nodes is the number of ways to choose $n + 1$ objects (the terminal nodes in a complete symmetric decision diagram of the function) from the r logic labels {0, 1, ... , r-1} with repetition, which is

$$\binom{n+r+1-1}{n+1} = \binom{n+r}{n+1}.$$

Q.E.D.

It is interesting to further compare the number of pseudo-voting functions with the number of symmetric functions on n variables and r logic values. Since the functional value of a symmetric function is the same no matter how the values are distributed among the variables, the number of such functions is the number of logic values, r, raised to the number of ways to select a group of logic values for the variables. Since the number of ways to choose r logic values for the n variables from the r possible values {0, 1, ... , r-1} with repetition is $\binom{n+r-1}{n}$, the number of multiple-valued symmetric functions on n variables and r values is $r^{\binom{n+r-1}{n}}$. When $r = 2$, this expression yields for the number of symmetric switching functions, 2^{n+1}. Therefore, from Theorem 1 it can be stated that,

Lemma 6 *The fraction of r-valued symmetric functions that have planar ROMDD's approaches* 0 *as n approaches infinity, where n is the number of variables.*

IV. AVERAGE NUMBER OF NODES IN ROMDD'S

Consider now the average number of nodes, $A_r(n)$ in ROMDD's of pseudo-voting functions. In a complete symmetric decision diagram of an r-valued pseudo-voting function on n variables, there are

$$1+2+...+(n+1)=\frac{(n+2)(n+1)}{2}$$

nodes. However, sequences of identical logic values yield nodes with identical children nodes that can be eliminated by the elimination rule. For example, Fig. 14 below shows how a group of three 1's and a group of two 3's reduce the node count (Fig. 14a) of a 4-valued 5-variable pseudo-voting function. Specifically, the group of three 1's results in the replacement of six nodes (dotted triangle) in the complete symmetric decision diagram of f by one node in the ROMDD (Fig. 14b) of f, while the group of two 3's results in the replacement of three nodes (dashed triangle) by one node.

In general, if there is a string of m identical logic values as labels of terminal nodes,

$$1+2+...+(m-1)+m=\frac{(m+1)m}{2}$$

nodes in the complete symmetric decision diagram are replaced by one node in the ROMDD.

The average number of nodes, $A_r(n)$ in ROMDD's of pseudo-voting functions is derived as follows,

$$A_r(n) \;=\; \frac{N_{\text{complete}} - N_{\text{reduction}}}{M_{\text{pseudo-voting}}},$$

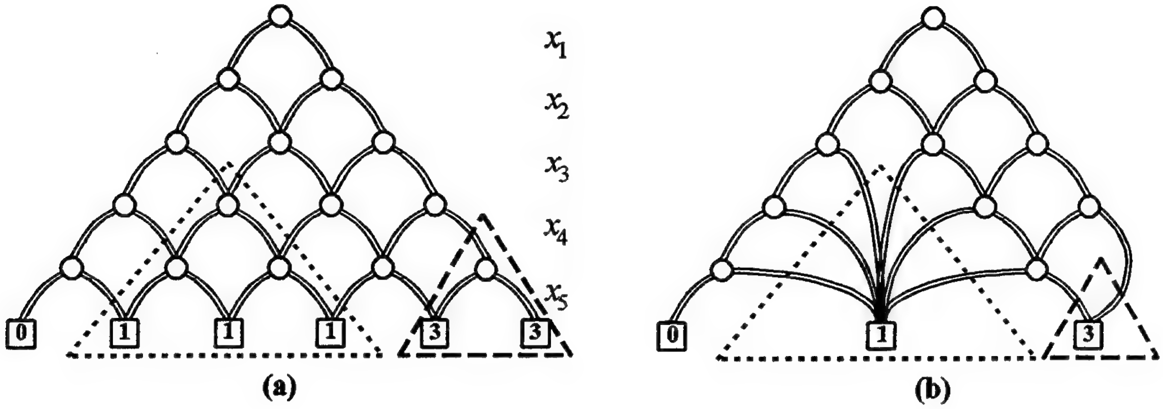


Figure 14. How groups of logic values reduce the nodes in OMDD's of pseudo-voting functions.

where N_{complete} is the total number of nodes in complete symmetric decision diagrams of pseudo-voting functions, and $N_{\text{reduction}}$ is the total reduction of nodes which occurs because of consecutive logic values on terminal nodes, and from Lemma 5, $M_{\text{pseudo-voting}}$ is the number of pseudo-voting functions. There are

$$M_{pseudo-voting} = (r-1)\binom{n+r}{n+1}$$

r-valued n-variable pseudo-voting functions, and, thus, this many complete symmetric decision diagrams. Therefore, the total number of nodes in complete symmetric decision diagrams of pseudo-voting functions is

$$N_{\text{complete}} = (r-1)\binom{n+r}{n+1}\frac{(n+2)(n+1)}{2}.$$

$N_{\text{reduction}}$ is calculated as follows. Any logic value can occur m times at the terminal nodes of a complete symmetric decision diagram, where $0 \le m \le n+1$. As shown previously, $m(m+1)/2$ nodes are replaced by a single node, yielding a reduction of $m(m+1)/2 - 1$ nodes.

There are

$$\binom{(r-1)+(n+1-m)-1}{(n+1-m)}$$

ways to choose a distribution of $r - 1$ remaining logic values to the $n + 1 - m$ remaining terminal nodes. Specifically, these are chosen by selecting $n + 1 - m$ objects (terminal nodes) from $r - 1$ objects (remaining logic values) with repetition. Since this is true for any of the r logic values and for any of the $r - 1$ ways to partition r logic values into two parts corresponding to labels on outgoing edges of each node, $N_{\text{reduction}}$ becomes,

$$N_{\text{reduction}} = \sum_{m=1}^{n+1} r(r-1)\left(\frac{m(m+1)}{2} - 1\right)\binom{(r-1)+(n+1-m)-1}{(n+1-m)}.$$

This sum is solved using generating functions. First, it is convenient to substitute $i = m - 1$. Doing this and rearranging yields,

$$N_{\text{reduction}} = \sum_{i=0}^{n}\left(\frac{r}{2}(r-1)\left(i^2+3i\right)\right)\binom{(r-1)+(n-i)-1}{(n-i)}.$$

A generating function $G(x)$ in which the coefficient of x^n in the above sum is

$$G(x) = A(x)B(x),$$

where the coefficient of x^i in $A(x)$ is

$$\frac{r}{2}(r-1)\left(i^2+3i\right)$$

and the coefficient of x^j in $B(x)$ is

$$\binom{(r-1)+j-1}{j}.$$

$A(x)$ can be calculated by observing that the generating function of i^2 is

$$\frac{x^2+x}{(1-x)^3},$$

while the generating function for i is

$$\frac{x}{(1-x)^2}.$$

To see this, differentiate both sides of $(1-x)^{-1} = 1 + x + x^2 + x^3 + \ldots$ This yields $(1-x)^{-2} = 1 + 2x + 3x^2 + \ldots$ Multiplying both sides by x yields the generating function for i. Differentiating both sides of this result and multiplying both sides by x yields the generating function for i^2. Therefore,

$$A(x) = r(r-1)\frac{2x-x^2}{(1-x)^3}.$$

The generating function for $B(x)$ is

$$B(x) = \frac{1}{(1-x)^{r-1}}.$$

Therefore,

$$G(x) = r(r-1)\frac{2x - x^2}{(1-x)^{r+2}}.$$

The coefficient of x^n in this expression is

$$N_{\text{reduction}} = r(r-1)\left[2\binom{(r+2)+(n-1)-1}{n-1} - \binom{(r+2)+(n-2)-1}{n-2}\right]$$

$$= r(r-1)\left[2\binom{n+r}{n-1} - \binom{n+r-1}{n-2}\right].$$

Now $A_r(n)$ can be calculated as,

Lemma 7 *The average number of nodes in ROMDD's of r-valued n-variable pseudo-voting functions is*

$$A_r(n) = \frac{\binom{n+r}{n+1}\left(\frac{(n+2)(n+1)}{2}\right) - r\left[2\binom{n+r}{n-1} - \binom{n+r-1}{n-2}\right]}{\binom{n+r}{n+1}}.$$

Consider the expression for $A_r(n)$ when n is large. $N_{\text{reduction}}$ can be written as

$$N_{\text{reduction}} = r(r-1)\left[2\frac{(n+r)(n+r-1)\ldots n}{(r+1)!} - \frac{(n+r-1)(n+r-2)\ldots(n-1)}{(r+1)!}\right].$$

When n is large compared to r, each term in the numerator is approximately n, and so, for large n, this expression is

$$N_{\text{reduction}} = r(r-1)\left(\frac{n^{r+1}}{(r+1)!}\right) \text{for large } n >> r.$$

Since $\binom{n+r}{n+1} = \frac{n^{r-1}}{(r-1)!}$ when n is large, $A_r(n)$ can be approximated as shown in Lemma 8 below.

Lemma 8 *The average number of nodes in ROMDD's of r-valued n-variable pseudo-voting functions for* $n \to \infty$ *is,*

$$A_r(n)_{n\to\infty} \approx n^2\left(\frac{1}{2} - \frac{1}{(r+1)}\right),$$

where f(n) $\approx$ *g(n) means* $\lim_{n\to\infty}\frac{f(n)}{g(n)} = 1.$

When $r = 2$, a pseudo-voting function is a conventional voting function and the average number of nodes is $n^2/6$. It is interesting to compare this result with the average number of nodes in the ROMDD's of r-valued n-variable symmetric functions. It is shown in [Ref. 5] that this number is $n^r/r!$, when n is large. That is, the average number of nodes in both cases is polynomial in n. However, the average number of nodes for general symmetric multiple-valued functions grows at a greater rate than the average for planar symmetric multiple-valued functions, suggesting that planarity restricts the number of nodes possible. It follows that the latter require less storage in computer representations.

V. WORST CASE NUMBER OF NODES IN ROMDD'S

In this chapter, the condition which causes the worst case number of nodes will be established, and an expression for the number of nodes will be derived.

When the average number of nodes was calculated in Chapter IV, all possible pseudo-voting functions and their ROMDD's were considered and counted to derive the expression for the average as stated in Lemma 7. For the worst case number of nodes, only one ROMDD has to be considered.

The idea of *node reduction* from Chapter IV can be applied here to calculate the worst case number of nodes in an ROMDD of an r-valued, n-variable pseudo-voting function. Before node reduction, a symmetric diagram is *complete*, and contains all nodes. The worst case number of nodes will involve the minimal node reduction possible within a complete symmetric diagram.

All r-valued, n-variable pseudo-voting functions can be represented by a complete symmetric decision diagram (OMDD) as shown in Fig. 14a. Node reduction converts the diagram into its reduced representation (ROMDD). All node reduction begins at the terminal nodes of a complete symmetric decision diagram, and with r less than the number of terminal nodes ($r < n + 1$), there will always be some node reduction.

The worst case number of nodes, $WC_r(n)$ is derived as follows.

Let m_i be the number of terminal nodes labeled by logic value i in a complete symmetric decision diagram. Thus, $\sum_{i=0}^{r-1} m_i = n+1$.

In preparation for the calculation of $WC_r(n)$, it is stated that,

Lemma 9 *The worst case number of nodes in an ROMDD of an r-valued, n-variable pseudo-voting function occurs when,* m_j-1 $\leq m_i \leq m_j$+1, *for all logic values i and j, such that* $0 \leq i < j \leq r$-1.

Proof On the contrary, assume there exists an i and j such that $m_j \leq m_i$ - 2. For example, the pseudo-voting function whose OMDD is represented in Fig. 14a, has the property, $m_j = m_i$ - 2 for i = 1 and j = 0.

Now calculate the reduction in nodes achieved when m_i terminal nodes are labeled by logic value i and m_j terminal nodes are labeled by logic value j, in a complete symmetric decision diagram.

For i, the reduction R_i is

$$R_i = (1+2+...+m_i) - 1 = \frac{(m_i+1)m_i}{2} - 1.$$

For j, the reduction R_j is

$$R_j = (1+2+...+m_j) - 1 = \frac{(m_j+1)m_j}{2} - 1.$$

Thus, the total reduction R_T becomes

$$R_T = R_i + R_j = \frac{(m_i+1)m_i}{2} - 1 + \frac{(m_j+1)m_j}{2} - 1$$

$$= \frac{m_i^2 + m_j^2 + m_i + m_j}{2} - 2. \qquad (\alpha)$$

Let $m_{avg} = (m_i + m_j)/2$. Assume $m_i \geq m_j$, and let $\Delta = (m_i - m_j)/2$. Then, $m_i = m_{avg} + \Delta$ and $m_j = m_{avg} - \Delta$. Substituting these into (α) yields

$$R_T = \frac{(m_{avg} + \Delta)^2 + (m_{avg} - \Delta)^2 + (m_{avg} + \Delta) + (m_{avg} - \Delta)}{2} - 2$$

$$= \frac{m_{avg}^{\;\;2} + m_{avg}^{\;\;2} + 2\Delta^2 + m_{avg} + m_{avg}}{2} - 2$$

$$= m_{avg}^{\;\;2} + \Delta^2 + m_{avg} - 2 .$$

With $(m_i + m_j)$ held constant for a given i and j, m_{avg} does not change, and the minimal reduction occurs when Δ is minimal. With $m_j \leq m_i - 2$, $\Delta \geq 1$. A smaller reduction (and thus a larger number of nodes) is achieved with a smaller Δ. Thus, $m_j \leq m_i - 2$ is not the worst case.

Q.E.D.

It follows from Lemma 9 that the minimum total reduction over all m_i and m_j is achieved with the most uniform distribution of logic values to terminal nodes. That is, the distribution of m_i's yielding the least reduction occurs when

$$m_i \approx \frac{n+1}{r}, \quad \text{as } n \to \infty, \quad \text{for } 0 \leq i \leq r-1 .$$

The total reduction for this worst case is

$$R_{T-WC} = r[(1+2+\ldots+m_i)-1] = r\left(\frac{\left(\frac{n+1}{r}+1\right)\left(\frac{n+1}{r}\right)}{2}-1\right) \approx r\left(\frac{\left(\frac{n}{r}\right)^2+\left(\frac{n}{r}\right)}{2}\right)_{n\to\infty} \approx r\left(\frac{n^2}{2r^2}\right)_{n\to\infty} = \frac{n^2}{2r}.$$

The total number of nodes before reduction is

$$N_T = 1+2+\ldots+(n+1) = \frac{(n+2)(n+1)}{2} \approx \frac{n^2}{2} \quad \text{for large } n.$$

Therefore,

Lemma 10 *The worst case number of nodes in an ROMDD of an r-valued, n-variable pseudo-voting function is*

$$WC_r(n) = N_T - R_{T-WC} \approx \frac{n^2}{2} - \frac{n^2}{2r} \approx \frac{n^2}{2}\left(1-\frac{1}{r}\right) \quad \textit{for large } n.$$

When $r = 2$,

$$WC_r(n) = \frac{n^2}{4} \quad \text{for large } n.$$

VI. PLANARITY OF FIBONACCI FUNCTIONS

The famous *Fibonacci* sequence (1, 1, 2, 3, 5, 8, ...) in which each term is the sum of the preceding two, occurs frequently in nature. Specifically, the n^{th} Fibonacci number F_n, is related as $F_n = F_{n-1} + F_{n-2}$, where $F_1 = F_2 = 1$. Leonardo Fibonacci (1170-1240) used it to describe the sizes of successive generations in an ideal rabbit population. From this sequence, the ancient Greeks derived the *Golden Ratio* as the convergence of the ratios of successive terms in the sequence. They used this ratio, $\frac{(1+\sqrt{5})}{2} \approx 1.6:1$, in proportioning their temples and public buildings.

The Fibonacci sequence has been a basis for extensive research for hundreds of years with an entire journal devoted to the subject, e.g. *The Fibonacci Quarterly*.

So far, this paper has considered symmetric functions. This chapter examines *Fibonacci* functions which are primarily non-symmetric threshold functions but nonetheless important and interesting. Some recent work has been performed in the *binary* decision diagram representation of Fibonacci functions [Ref. 14]. This chapter shows necessary and sufficient conditions for planarity in the ROMDD representation of multiple-valued variable, two-valued Fibonacci functions.

Lemma 11 *The sum of the terms in a Fibonacci sequence is related as,*

$$\sum_{i=1}^{n} F_i = F_{n+2} - 1 \text{ for } n \geq 1. \tag{1}$$

Proof *(by induction)* For $n = 1$, $\sum_{i=1}^{n} F_i = F_1$ and $F_{n+2} - 1 = F_3 - 1 = 2 - 1 = 1$.

Assume that $\sum_{i=1}^{n} F_i = F_{n+2} - 1$ is true for $n = m$. It is then shown that the expression is true for $n = m + 1$.

Consider $\sum_{i=1}^{m+1} F_i$, which can be expressed as,

$$\sum_{i=1}^{m+1} F_i = \sum_{i=1}^{m} F_i + F_{m+1}, \tag{2}$$

where, from the *inductive assumption*,

$$\sum_{i=1}^{m} F_i = F_{m+2} - 1. \tag{3}$$

Substituting (3) into (2) yields

$$\sum_{i=1}^{m+1} F_i = F_{m+2} - 1 + F_{m+1}. \tag{4}$$

However, from the Fibonacci recurrence relation, $F_{m+2} + F_{m+1} = F_{m+3}$ and (4) becomes

$$\sum_{i=1}^{m+1} F_i = F_{m+3} - 1,$$

which is (1) with $n = m + 1$. This proves the hypothesis.

Q.E.D.

Definition 7 *A Fibonacci function is a threshold function with weight-threshold vector* $(F_n, F_{n-1}, F_{n-2}, \ldots, F_2, F_1; T)$*, where* F_i *is the* i^{th} *Fibonacci number and the threshold* (T) *is in the range* $0 < T < F_{n+2}$ *for binary-valued function variables and in the range* $0 < T \leq (r - 1)[F_{n+2} - 1]$ *for multiple-valued function variables.*

Lemma 12 *The ROBDD of a Fibonacci function with variables ordered* $x_1, x_2, \ldots, x_{n-1}$*, and* x_n *is planar for any n.*

Proof [Ref. 14] shows a construction of the ROBDD of a Fibonacci function using three types of structures. Each structure and its relation to other structures is planar. Thus, the ROBDD of a Fibonacci function is planar.

Q.E.D.

Fibonacci functions with multiple-valued variables are an interesting extension to the binary case. A necessary and sufficient condition for planarity in the ROMDD's of such functions is derived for $r > 2$. The demonstration begins with a definition of *maximum weighted sum*.

Definition 8 *The maximum weighted sum (MWS) of a Fibonacci function is*

$$MWS = (r-1)\sum_{i=1}^{n} F_i .$$

The following expression evolves from Definition 8 and Lemma 11,

$$MWS = (r-1)\sum_{i=1}^{n} F_i = (r-1)[F_{n+2} - 1] .$$

From Fig. 15, it can be seen that *MWS* has a graphical interpretation in the ROMDD of a Fibonacci function. Specifically, it is the cumulative weight associated with the path from the root node to the 1 terminal node where $x_1 = x_2 = \ldots = x_n = r - 1$.

The demonstration proceeds from the "bottom-up." First, the maximum number of nodes at the lowest (x_n) level is derived. This derivation shows that the x_n level has more than one node under certain conditions. Next, it is shown how more than one node at the x_n level causes crossings and thus non-planarity. Finally, the conditions for planarity are established.

Lemma 13 *Let f be a two-valued Fibonacci function with* $r > 2$ *and* $n > 1$. *For any given threshold (T), the ROMDD of f has a maximum of* $r - 1$ *nodes at the* x_n *level. This maximum number of nodes occurs when T is in the range,* $(r - 1) \leq T \leq MWS - (r - 2)$. *Otherwise, there are incrementally one less node, reaching a minimum of one, at the* x_n *level as T decreases from* $(r-2) \rightarrow 1$ *or increases from* $[MWS - (r-3)] \rightarrow MWS$.

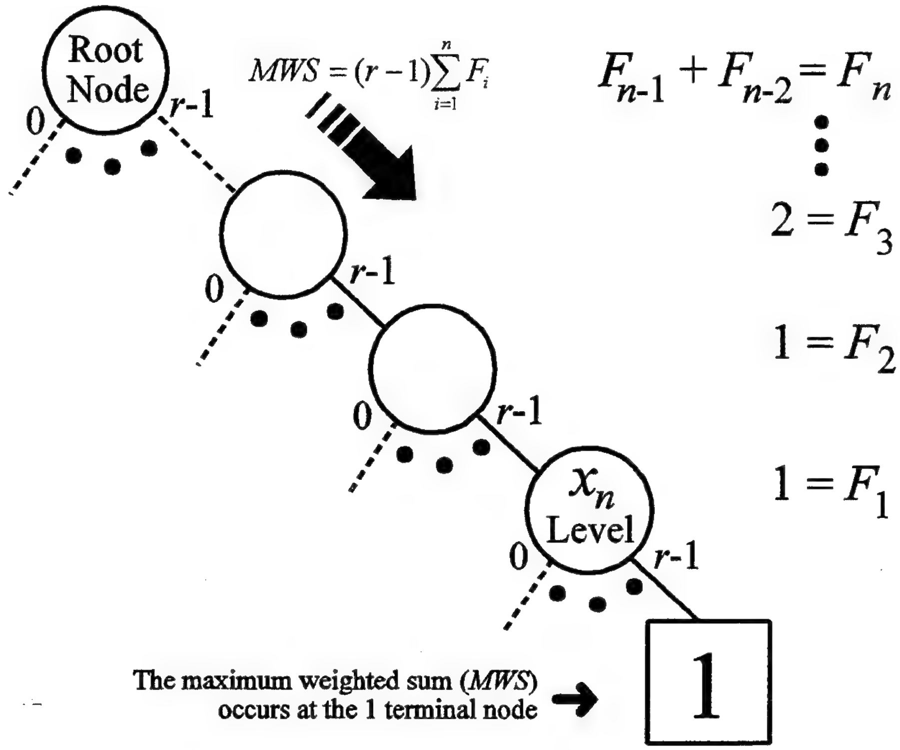


Figure 15. A partial ROMDD of a Fibonacci function showing how *MWS* is achieved.

Proof Every node at the x_n level has an edge labeled 0 that must go to the 0 terminal node and an edge labeled r - 1 that must go to the 1 terminal node. The proof of this is similar to the proof of Lemma 1.

Since no two edges originating from the same node may cross, there must be an a such that edges from any node η at the x_n level labeled 0, 1, ..., a go to the 0 terminal node and edges labeled a + 1, a + 2, ..., r - 1 go to the 1 terminal node. It is known that $0 \leq a \leq r - 2$; therefore, there are only r - 1 possible values for a. No two nodes can have the same a (by the Merging Rule, these two nodes would be merged). Thus, there are at most r - 1 nodes at the x_n level.

The Fibonacci weight (F_1) at the lowest level(x_n) of the ROMDD is always 1, so x_n contributes only 0, 1, ..., or r - 1 to the weighted sum, as shown in Fig. 16.

The cumulative weights (CW) associated with nodes at the x_n level are T - 1, T - 2, ..., T - (r - 1) as shown in Fig. 16. A CW outside this range is never achievable at the x_n level because the maximum contribution by $x_n = r - 1$ cannot exceed the threshold with $CW < T - (r - 1)$, and the threshold will already be met with $CW > T - 1$. This causes the x_n level to be "skipped" with edges proceeding directly to the 0 and 1 terminal nodes, respectively. Therefore, a CW is achievable in the range, $T - (r - 1) \leq CW \leq T - 1$ dependent upon the chosen T.

If $1 \leq T < r - 1$, then each CW at the x_n level must be in the range, $0 \leq CW \leq T - 1$ because $\min(CW) \geq 0$ at the x_n level as T decreases to 1. This range for T causes $((T - 1) - 0) + 1 = T$ distinct values for CW and thus T distinct nodes at the x_n level for $1 \leq T < r - 1$.

If $MWS - (r - 2) < T \leq MWS$, then each CW at the x_n level must be in the range, $T - (r - 1) \leq CW \leq MWS - (r - 1)$ because $\max(CW) = MWS - (r - 1)$ at the x_n level as T increases to the maximum weighted sum (MWS). The expression for $\max(CW)$ at the x_n level is attributable to the successive contributions of $x_1 = x_2 = \ldots = x_{n-1} = r - 1$ which result in a value at the x_n level that is r - 1 less than the MWS because $F_1 = 1$ (see Fig. 15). This range for T causes $[(MWS - (r - 1)) - (T - (r - 1))] + 1 = MWS - T + 1$ distinct values for CW and thus $MWS - T + 1$ distinct nodes at the x_n level for $MWS - (r - 2) < T \leq MWS$.

Now for the remaining range of T, $(r - 1) \leq T \leq MWS - (r - 2)$, the entire range of CW at the x_n level, $T - (r - 1) \leq CW \leq T - 1$, is achievable because $\min(CW) \geq 0$ and $\max(CW) \leq$

MWS - (r - 1) both hold for this range of T. This range of T causes $(T - 1) - (T - (r - 1)) + 1 = r - 1$ distinct values for CW and thus r - 1 distinct nodes at the x_n level.

Since r - 1 is greater than T for $1 \leq T < r - 1$ and greater than $MWS - T + 1$ for $MWS - (r - 2) < T \leq MWS$ because $r > 2$, the maximum number of nodes at the x_n level is r - 1 when T is in the range, $(r - 1) \leq T \leq MWS - (r - 2)$.

Q.E.D.

From the proof of Lemma 13, it is shown that a two-valued Fibonacci function with $r > 2$ and $n > 1$ has T distinct nodes at the x_n level for $1 \leq T < r - 1$ and $MWS - T + 1$ distinct nodes at the x_n level for $MWS - (r - 2) < T \leq MWS$. These ranges for T show that there is more than one node at the x_n level unless $T = 1$ or $T = MWS$. From Lemma 1, two or more nodes at the x_n level (associated with the lowest variable) creates at least one crossing, thus non-planarity. Therefore,

Theorem 2 *Let f be a two-valued Fibonacci function with* $r > 2$ *and* $n > 1$. *f is planar iff* $T = 1$ *or* T = *Maximum Weighted Sum (MWS).*

From Lemma 3, similar results were obtained for binary voting functions. Specifically, the restrictions placed on the value of T to obtain planarity in Lemma 3 and Theorem 2 represent the AND and OR functions as shown in Fig. 5.

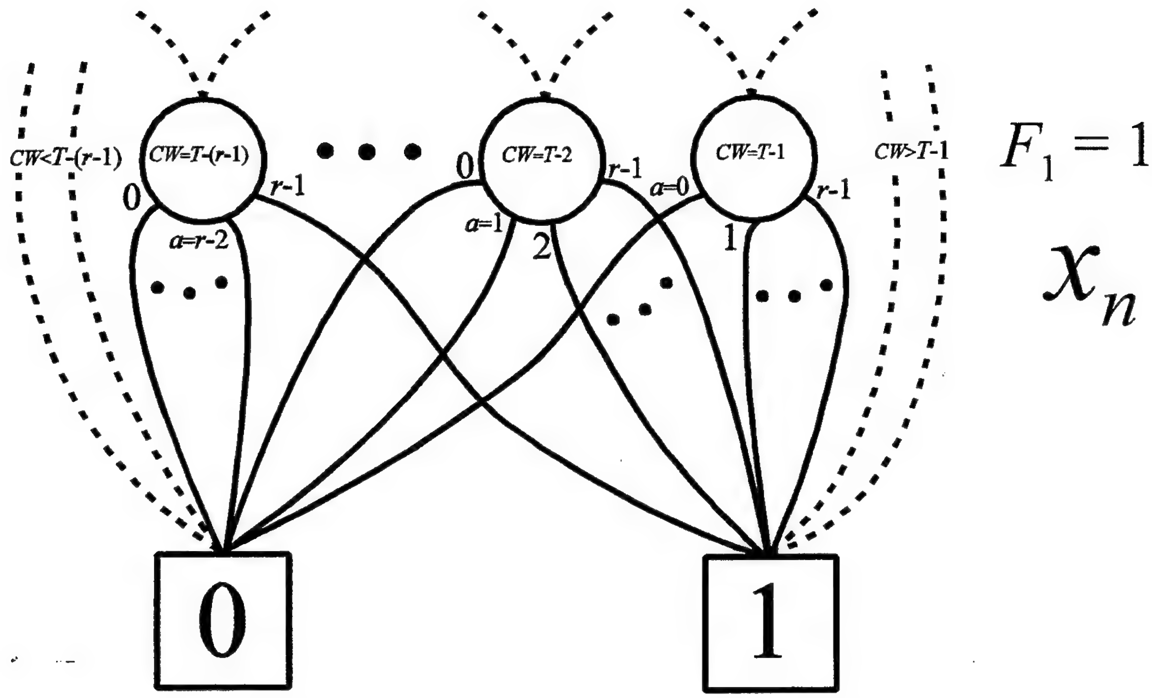


Figure 16. The x_n level of an ROMDD of a Fibonacci function.

VII. CONCLUSION

In this thesis, the planarity of ROMDD's of multiple-valued symmetric functions has been considered. The main result is that the ROMDD of a symmetric multiple-valued function f is planar if and only if f is a pseudo-voting function. A major source of delay in VLSI is interconnect. Planarity in ROMDD's reduces delay in digital circuits, an important consideration in their design, by preventing crossings among interconnect in VLSI. Insights gained from this facilitated the calculation of the average and worst case number of nodes in planar ROMDD's of r-valued symmetric functions on n variables. It was shown that the average number of nodes for general symmetric multiple-valued functions grows at a greater rate than the average for planar symmetric multiple-valued functions, suggesting that planarity restricts the number of nodes possible. It follows that the latter require less storage in computer representations.

Other results include a characterization of threshold values for which a two-valued voting function on r-valued variables is planar. A similar result is obtained for the unique class of two-valued Fibonacci functions with r-valued variables.

An outcome of this work is the observation that the fraction of symmetric functions that are planar approaches 0 as the number of variables increases for any radix $r \geq 2$. It is fully expected that this is true of the general functions; that is, it is conjectured that the fraction of multiple-valued functions which have planar ROMDD's approaches 0 as the number of variables approaches infinity. This suggests that planar ROMDD's are rare among

all multiple-valued functions. However, important functions indeed have planar ROMDD's, e.g. AND, OR, and general voting functions.

The results can be extended in a number of ways. Restriction 1 has allowed specific statements to be made about the planarity of a class of functions. Allowing other permutations of edge assignments and/or terminal node assignments enlarges the class of functions with planar ROMDD's considerably. This class can be enlarged further by allowing unary functions along the edges. That is, two nodes can be combined if their function differs by a mapping among function (output) values. In binary, such mappings are described as complemented edges.

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